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On the quantum harmonic oscillator driven by a strong linearly polarized field

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Abstract. We present an approximate analytic solution to the time-dependent Schrödinger equation of a charged harmonic oscillator in the presence of a strong laser field, based upon the technique advanced recently by Frasca. The resulting wavefunction is then used to derive expressions for the survival probability of the ground state and the probability of transition to an arbitrary higher state.

1. Introduction

Numerical integration of the time-dependent Schrödinger equation in space and time, made possible by the availability of powerful computing techniques, is by far the most reliable means of extracting information about the evolution of quantum mechanical systems in the presence of strong time-dependent perturbations [1]. Standard perturbation theory has had very little success in this area.

By introducing a change of time scale, Frasca [2] has recently advanced an asymptotic series solution to the time-dependent Schrödinger equation of a strongly perturbed quantum system. The resulting series for the wavefunction of the system is basically an interaction picture series with the roles of the time-dependent perturbation and unperturbed Hamiltonian interchanged. In the same publication, Frasca demonstrated the usefulness of his technique by treating a few simple examples. He later employed the same procedure [3] to find the wavefunction of a quantum harmonic oscillator subjected to the perturbation $V(x,t) = q\phi\cos(kx - \omega t)$, where q is the electric charge and ϕ is the electrostatic potential. In this example, Frasca shows that, to order zero, the oscillator ground-state wavefunction evolves into a superposition of coherent states.

In this paper, a slightly modified model of the system just described is considered. Instead of the spatio-temporal perturbation, we consider an oscillating electric field polarized parallel to the direction of motion of the charged oscillator and with a large and uniform amplitude. We evaluate the time-evolved wavefunction to order two as opposed to Frasca's zero-order calculation. We then employ the approximate wavefunction in a calculation of the survival probability of the ground state and the probability of transition to the nth oscillator state, all as functions of the time.

Our choice of the harmonic oscillator is motivated by its simplicity. It is hoped that its study will give us important clues on how a more complicated system with a few bound states and a continuum, like a model ion or atom, may be treated. Conclusions on how the wavefunction of the oscillator will evolve in time as a result of the time-dependent perturbation, may motivate researchers to apply the same program to real atoms, ions and

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molecules under similar conditions. From the time-evolved wavefunction, observables may be calculated which may subsequently be compared with experimental findings.

The choice is further motivated by the fact that an exact solution to the problem is possible employing the interaction picture [4]. We also develop this solution and compare it with the approximate solution based upon Frasca's approach.

The subject of the present paper may also prove to be very useful in understanding the classical problem of stochastic heating [5]. The example we study in this paper represents the quantum analogue of such a problem. Furthermore, the Mössbauer effect [6] can be thought of as the three-dimensional version of precisely this example. In the Mössbauer effect the oscillator is a nucleus vibrating about its equilibrium position in a crystal. Absorption, by the nucleus, of the gamma photons takes place when the field frequency matches the frequency of transition from an initial state to one of the higher oscillator states.

The rest of the paper is organized as follows. In the next section the technique advanced recently by Frasca for the approximate solution of a strongly-perturbed system will be reviewed. This will be used in section 3 to calculate the time-evolved wavefunction of a quantum harmonic oscillator. The exact solution to the problem will also be presented in the same section. Section 4 will be devoted to the derivation, using Dirac oscillator creation and annihilation operators, of an expression for the probability that the oscillator ground state will survive the effect of the intense laser field after it has been turned on for time t. The probability of transition from the ground state to higher states will also be calculated under similar conditions. In the last section, the approximate solution will be compared indirectly with the exact one and some comments and conclusions will be given.

2. The theory

Frasca's original proposal [2] to solve approximately the time-dependent Schrödinger equation

$$[H_0 + \epsilon V(t)]\psi = i\hbar \frac{\partial \psi}{\partial t} \tag{1}$$

is based upon the change of time scale $t \to t/\epsilon$ and the subsequent substitution

$$\psi(t) = \sum_{k=0}^{\infty} \frac{\psi^{(k)}}{\epsilon^k}. \tag{2}$$

In equation (1), H_0 is the unperturbed (free) Hamiltonian and $V(t)$ is the perturbation. ϵ is introduced here as a development parameter and is assumed to be large in order to ensure convergence of the series (2). When the substitution (2) is used in equation (1), together with the suggested change of time scale, the following set of pairwise-coupled differential equations emerges, as one makes an order by order comparison

$$V\psi^{(0)} = i\hbar \frac{\partial \psi^{(0)}}{\partial t} \tag{3}$$

$$H_0\psi^{(0)} + V\psi^{(1)} = i\hbar \frac{\partial \psi^{(1)}}{\partial t} \tag{4}$$

$$\vdots$$

$$H_0\psi^{(k-1)} + V\psi^{(k)} = i\hbar \frac{\partial \psi^{(k)}}{\partial t}. \tag{5}$$

The order zero equation can be integrated formally and yields $\psi^{(0)}(t)$. When $\psi^{(0)}(t)$ is then used in the equation of order one, it too admits a formal solution $\psi^{(1)}(t)$. This process can, in principle, be continued and, to kth order, one gets

$$\psi^{(k)}(t) = \left(-\frac{\mathrm{i}}{\hbar}\right)^k T_0(t) \int_0^t H_0'(t_1)\,\mathrm{d}t_1 \int_0^{t_1} H_0'(t_2)\,\mathrm{d}t_2 \ldots \int_0^{t_{k-1}} H_0'(t_k)\,\mathrm{d}t_k \psi(0) \tag{6}$$

where

$$T_0(t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t V\left(\frac{t'}{\epsilon}\right)\mathrm{d}t'\right] \tag{7}$$

and

$$H_0'(t) = T_0(t)^{-1} H_0 T_0(t) \tag{8}$$

and $\psi(t) = \psi(x,t), \ldots$ etc. The assumption common to all the equations above is that the perturbing field is turned on at $t = 0$ while the system is in the state $\psi(0)$ belonging to the domain of the unperturbed Hamiltonian, namely

$$H_0\psi(0) = E\psi(0). \tag{9}$$

As is evident from equations (6)–(8), the technique is effectively an interaction picture [7] with the roles of the perturbation and unperturbed Hamiltonian interchanged.

3. The time-evolved oscillator wavefunction

We study a charged harmonic oscillator of mass m, charge e and frequency ω_0, prepared in its ground state and subjected, beginning at $t = 0$, to an oscillating electric field $\mathcal{E} = \mathcal{E}_0\cos(\omega t)$, polarized in the direction of motion of the oscillator. $\mathcal{E}_0$ is the field amplitude, assumed large and uniform. This monochromatic plane wave of frequency ω is often used to model ideally the electric component of the radiation field of a laser.

The free Hamiltonian of the oscillator is given by [8]

$$H_0 = \hbar\omega_0(a^\dagger a + \tfrac{1}{2}) \tag{10}$$

where $a^\dagger$ and a are the creation and annihilation operators, respectively. In the dipole approximation, the perturbation is given by

$$V(t) = -e\mathcal{E}_0 x\cos(\omega t) = -e\mathcal{E}_0\sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger)\cos(\omega t). \tag{11}$$

We first develop the exact solution to the time-dependent Schrödinger equation of the given system. This is done in the following subsection.

3.1. The exact solution

Assume that the system has been prepared in its ground state $|0\rangle$ at $t = 0$, at which instant the perturbation, written as $V(x, t) = \hbar F(t)(a + a^\dagger)$, is turned on. An exact solution to the time-dependent Schrödinger equation in the interaction picture exists [4]. According to it, the system evolves in time into the coherent state $|\beta\rangle$ the parameter of which is given as a function of the time by

$$\begin{aligned}\beta(t) &= -\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}\omega_0(t-t')}F(t')\,\mathrm{d}t' \\ &= \frac{e\mathcal{E}_0}{\sqrt{2m\omega_0\hbar}}\frac{\omega_0(\cos\omega t - \cos\omega_0 t) + \mathrm{i}(\omega\sin\omega t - \omega_0\sin\omega_0 t)}{\omega_0^2 - \omega^2}\end{aligned} \tag{12}$$

or

$$\beta(\tau) = \frac{1}{2}r\sqrt{q}\frac{(\cos\tau - \cos 2\tau/r) + \mathrm{i}\left(\frac{1}{2}r\sin\tau - \sin 2\tau/r\right)}{1 - \left(\frac{1}{2}r\right)^2}. \tag{13}$$

The parameters q and r, in equation (13), will be defined below and $\tau = \omega t$.

3.2. The approximate solution

We now apply the technique outlined in section 2 to the same system. The result will be an approximate analytic wavefunction for the quantum harmonic oscillator. In the presence of the exact solution developed above, the approximate wavefunction is, of course, not very useful. Nevertheless, since we are looking for clues on how real atomic systems behave under similar conditions, for which no exact analytic solutions exist, the pursuit of an approximate solution for the simple system of a driven harmonic oscillator, remains desirable. Now, equations (6) and (7) yield

$$\begin{aligned}|\psi^{(0)}\rangle &= \exp\left\{\mathrm{i}\frac{e\mathcal{E}_0}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger)\sin\omega t\right\}|0\rangle \\ &= \mathrm{e}^{\mathrm{i}\alpha(t)(a+a^\dagger)}|0\rangle \\ &= |\mathrm{i}\alpha\rangle\end{aligned} \tag{14}$$

where

$$\alpha(t) = \frac{e\mathcal{E}_0}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega_0}}\sin(\omega t). \tag{15}$$

Recall that the Glauber [9] coherent state has the following number representation

$$|\mathrm{i}\alpha\rangle = \mathrm{e}^{\mathrm{i}\alpha(a^\dagger + a)}|0\rangle = \mathrm{e}^{-|\mathrm{i}\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\mathrm{i}\alpha)^n}{\sqrt{n!}}|n\rangle. \tag{16}$$

In order to move on to the next level of approximation, we need to evaluate the following operator, using well known identities [10]

$$\begin{aligned}H_0' = T_0^{-1}H_0T_0 &= \hbar\omega_0\left[\mathrm{e}^{-\mathrm{i}\alpha(a+a^\dagger)}(a^\dagger a + \tfrac{1}{2})\mathrm{e}^{\mathrm{i}\alpha(a+a^\dagger)}\right] \\ &= H_0 + \hbar\omega_0\left[\alpha^2 - \mathrm{i}\alpha(a - a^\dagger)\right].\end{aligned} \tag{17}$$

The time dependence in equation (17) is all in $\alpha(t)$. Assuming that the system is prepared initially (at $t = 0$) in its ground state $|\psi(0)\rangle \equiv |0\rangle$, the following integrals will be needed

$$\int_0^t \alpha^2(t')\,\mathrm{d}t' = \frac{\hbar}{2m\omega_0}\left(\frac{e\mathcal{E}_0}{\hbar\omega}\right)^2 \frac{1}{4\omega}[2\omega t - \sin(2\omega t)] \tag{18}$$

$$\int_0^t \alpha(t')\,\mathrm{d}t' = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\frac{e\mathcal{E}_0}{\hbar\omega}\right)\frac{2}{\omega}\sin^2\left(\tfrac{1}{2}\omega t\right). \tag{19}$$

Before moving on to the explicit construction of $|\psi^{(1)}\rangle$, according to equation (6), let us introduce a few dimensionless parameters, borrowed from the general literature on laser–atom interactions. Those parameters are $q \equiv E_\mathrm{q}/E_0$ and $r \equiv E_\mathrm{ph}/E_0$, where $E_\mathrm{q} = (e\mathcal{E}_0)^2/4m\omega^2$ is the average classical energy of oscillation (quiver energy) of the particle in the given field, E_0 is the binding energy (in our case E_0 is the ground-state energy $\frac{1}{2}\hbar\omega_0$) and E_ph is the photon energy of the laser field. In terms of these parameters, equation (17) takes the form

$$H_0'(t) = (H_0 + E_\mathrm{q}) - E_\mathrm{q}\cos(2\omega t) + \mathrm{i}\,2\sqrt{E_\mathrm{q}E_0}\,\sin(\omega t)(a^\dagger - a). \tag{20}$$

With the help of equations (18)–(20), the integrals needed to evaluate the first few corrections to the oscillator ground-state wavefunction turn out to be straightforward. For example,

$$\int_0^{t_{k-1}} H_0'(t_k)\,\mathrm{d}t_k = (H_0 + E_\mathrm{q})t_{k-1} - E_\mathrm{q}\frac{\sin(2\omega t_{k-1})}{2\omega} + \mathrm{i}\frac{4}{\omega}\sqrt{E_\mathrm{q}E_0}\sin^2\left(\tfrac{1}{2}\omega t_{k-1}\right)(a^\dagger - a) \tag{21}$$

for all $k \geqslant 1$. Equations (20) and (21) are central to the calculation of the wavefunction to all orders. Hence, with the help of (21) one arrives without difficulty at

$$|\psi^{(1)}\rangle = \frac{\mathrm{e}^{\mathrm{i}\alpha(a+a^\dagger)}}{r}\left\{4\sqrt{q}\sin^2\left(\tfrac{1}{2}\omega t\right)(a^\dagger - a) + \mathrm{i}\left[\tfrac{1}{2}q\sin(2\omega t)(q+1)(\omega t)\right]\right\}|0\rangle. \tag{22}$$

Also, multiplying the right-hand sides of (20) and (21) and carrying out another time integration, we get the second-order correction to the wavefunction. The algebra involved is quite elaborate. We give the result below without the details of the calculation.

$$\begin{aligned}|\psi^{(2)}(t)\rangle = \frac{\mathrm{e}^{\mathrm{i}\alpha(a^\dagger+a)}}{r^2}&\left\{\left[\tfrac{1}{2}q(q+1)(\omega t)\sin(2\omega t) - \tfrac{1}{2}(q+1)^2(\omega t)^2 - \tfrac{1}{8}q^2\sin^2(2\omega t)\right.\right.\\ &\left. + 8q\sin^4(\tfrac{1}{2}\omega t)(a^\dagger - a)^2\right] + \mathrm{i}2\sqrt{q}\left[\sin^2(\tfrac{1}{2}\omega t)[q\sin(2\omega t) - 2(q+1)(\omega t)]\right.\\ &\left.\left.\times(a^\dagger - a) + 2[\sin(\omega t) - (\omega t)](a^\dagger + a)\right]\right\}|0\rangle. \end{aligned}\tag{23}$$

We proceed to express the time-evolved wavefunction in terms of coherent states. This will be followed by a calculation of the survival probability of, and the probability of transition from, the ground state.

Putting (14), (22) and (23) together, the approximate time-evolved wavefunction may be written in terms of the coherent state $|\mathrm{i}\alpha\rangle$ as

$$
\begin{aligned}
|\psi(\tau)\rangle = \Big\{ 1 + \frac{1}{r}\left[4\sqrt{q}\sin^2\left(\frac{\tau}{2}\right)(a^\dagger + \mathrm{i}\alpha) + \mathrm{i}[\tfrac{1}{2}q\sin 2\tau - (q+1)\tau]\right] \\
+ \frac{1}{r^2}\Big[\tfrac{1}{2}q(q+1)\tau\sin 2\tau - \tfrac{1}{2}(q+1)^2\tau^2 - \tfrac{1}{8}q^2\sin^2 2\tau \\
+ 8q\sin^4(\tfrac{1}{2}\tau)(a^\dagger - a + 2\mathrm{i}\alpha)^2 + \mathrm{i}2\sqrt{q}\Big[\sin^2(\tfrac{1}{2}\tau)[q\sin 2\tau - 2(q+1)\tau] \\
+ 2(\sin\tau - \tau)\Big](a^\dagger + \mathrm{i}\alpha)\Big] + \mathcal{O}\left(\frac{1}{r^3}\right)\Big\}|\mathrm{i}\alpha\rangle
\end{aligned} \tag{24}
$$

where $\tau \equiv \omega t$.

4. Applications

Using the well known properties of the coherent states [10], the approximate time-evolved oscillator state function may be used to calculate any observable pertraining to the system. In particular, the survival probability (or population) of the oscillator ground state can be calculated as the square of the modulus of the following amplitude

$$
\begin{aligned}
\mathcal{A}_0(\tau) = \langle 0|\psi\rangle = \exp(-\tfrac{1}{2}q\sin^2\tau)\Big\{1 + \frac{\mathrm{i}}{r}\left[4q\sin\tau\sin^2\tfrac{1}{2}\tau + \tfrac{1}{2}q\sin 2\tau - (q+1)\tau\right] \\
+ \frac{1}{r^2}\Big[\tfrac{1}{2}q(q+1)\tau\sin 2\tau - \tfrac{1}{2}(q+1)^2\tau^2 \\
- \tfrac{1}{8}q^2\sin^2 2\tau - 8q(1 + q\sin^2\tau)\sin^4\tfrac{1}{2}\tau \\
- 2q\sin\tau\left[\sin^2\tfrac{1}{2}\tau[q\sin 2\tau - 2(q+1)\tau] + 2(\sin\tau - \tau)\right]\Big] + \mathcal{O}\left(\frac{1}{r^3}\right)\Big\}.
\end{aligned} \tag{25}
$$

This is to be compared with the exact amplitude

$$
\langle 0|\beta(\tau)\rangle = \exp[-\tfrac{1}{2}|\beta(\tau)|^2]. \tag{26}
$$

In figure 1, we show the ground-state population as a function of the time over approximately 16 field cycles. Note that agreement between the predictions of the exact solution (26) and the approximate one (25) is almost complete over approximately 10 field cycles.

On the other hand, projecting the nth oscillator state onto $|\psi\rangle$, we obtain the following amplitude

$$
\begin{aligned}
\mathcal{A}_{0\to n}(\tau) = \langle n|\psi\rangle = \frac{(\mathrm{i}\alpha)^n}{\sqrt{n!}}\exp\left(-\tfrac{1}{2}q\sin^2\tau\right) \\
\times\Big\{1 + \frac{\mathrm{i}}{r}[\tfrac{1}{2}q\sin 2\tau - (q+1)\tau - 2(n - q\sin^2\tau)\tan\tfrac{1}{2}\tau] + \frac{1}{r^2} \\
\times\Big[\tfrac{1}{2}q(q+1)\tau\sin 2\tau - \tfrac{1}{2}(q+1)^2\tau^2 - \tfrac{1}{8}q^2\sin^2 2\tau + 8q\sin^4\tfrac{1}{2}\tau \\
\times\left[2n - 1 - q\sin^2\tau - \frac{n(n-1)}{q\sin^2\tau}\right] + \tan\tfrac{1}{2}\tau(n - q\sin^2\tau) \\
\times[q\sin 2\tau - 2(q+1)\tau] + 2(\sin\tau - \tau)\Big] + \mathcal{O}\left(\frac{1}{r^3}\right)\Big\}.
\end{aligned} \tag{27}
$$

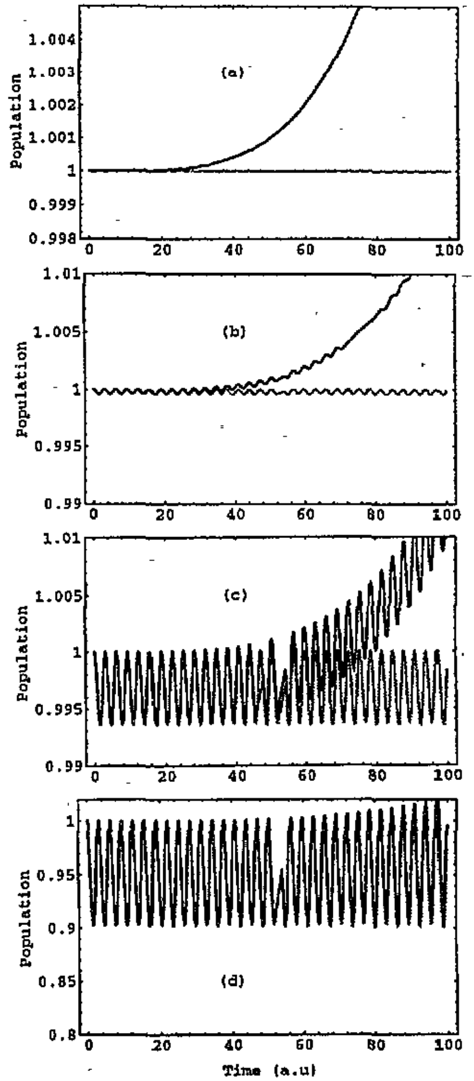

Figure 1. The ground-state population plotted over approximately 16 field cycles. Note that agreement between the exact solution (light curve) and the approximate one (bold curve) is almost complete over 6–10 field cycles. The peak field strengths used are: (a) 0.01, (b) 0.04, (c) 0.16 and (d) 0.64 au. For all plots $\omega_0 = 0.01$ and $\omega = 1$ au.

The square of the modulus of $\mathcal{A}_{0\to n}(\tau)$ gives the probability of transition from the initial ground state to the nth state at time t. Note that $\mathcal{A}_{0\to n}(\tau)$ reduces to $\mathcal{A}_0(\tau)$ in the appropriate limit ($n \to 0$), as expected. Close inspection of equation (27) reveals also that the leading term in $\mathcal{A}_{0\to n}$ gives the following transition probability

$$\mathcal{P}_{0\to n}(\tau) = \frac{(q \sin^2 \tau)^n}{n!} \exp(-q \sin^2 \tau) \tag{28}$$

a Poissonian with mean and variance equal to $\alpha^2 = q \sin^2 \tau$, characteristic of a coherent state. On the other hand, the exact amplitude corresponding to $\mathcal{A}_{0\to n}(\tau)$ is

$$\langle n|\beta(\tau)\rangle = \frac{\beta^n}{\sqrt{n!}}\exp(-\tfrac{1}{2}|\beta|^2). \tag{29}$$

This corresponds to the transition probability

$$P_{0\to n}^{\text{exact}} = \frac{(|\beta|^2)^n}{n!}\mathrm{e}^{-|\beta|^2} \tag{30}$$

a Poissonian with mean and variance equal to $|\beta|^2$. Note that for high-field frequencies, $\omega \gg \omega_0$, the terms involving ω_0 in both numerator and denominator of equation (12) may be dropped. This leaves us with the following limit

$$\beta(t) \to \mathrm{i}\sqrt{q}\sin\omega t = \mathrm{i}\alpha(t) \tag{31}$$

and equations (28) and (30) become identical. Thus, at least for high-field frequencies, the approximate and exact solutions give similar transition probability distributions.

5. Discussion and conclusions

Our main concern in this paper has been the application of a new perturbative technique for the analytic treatment of a strongly perturbed quantum system, the harmonic oscillator. Fortunately, such a system admits an exact analytic solution. How well the approximate solution agrees with the exact one can be found out from expanding $|\beta(\tau)\rangle$ and comparing the resulting series with $|\psi(\tau)\rangle$, term by term. This may be possible but is certainly far from being straightforward. Instead, a graphical comparison of the exact and approximate ground state populations may be sufficient. This is shown in figure 1. Furthermore, in order to determine the time region over which the approximate solution approaches the exact one closely enough, we will calculate the probability that the approximate solution will be the coherent state $|\beta(\tau)\rangle$. In other words, we will calculate the square of the modulus of the following inner product

$$\begin{aligned}\langle\beta(\tau)|\psi(\tau)\rangle &= \exp[-(\tfrac{1}{2}\alpha^2 + \tfrac{1}{2}|\beta|^2 - \mathrm{i}\alpha\beta^*)] \\ &\quad\times\left\{1 + \frac{1}{r}\left[4\sqrt{q}\sin^2(\tfrac{1}{2}\tau)(\beta^* + \mathrm{i}\alpha) + \mathrm{i}[\tfrac{1}{2}q\sin 2\tau - (q+1)\tau]\right]\right. \\ &\quad + \frac{1}{r^2}\Big[\tfrac{1}{2}q(q+1)\tau\sin 2\tau - \tfrac{1}{2}(q+1)^2\tau^2 - \tfrac{1}{8}q^2\sin^2 2\tau \\ &\quad + 8q\sin^4\tfrac{1}{2}\tau(\beta^{*2} - 1 - \alpha^2 + 2\mathrm{i}\alpha\beta^*) \\ &\quad + \mathrm{i}2\sqrt{q}\left[\sin^2\tfrac{1}{2}\tau[q\sin 2\tau - 2(q+1)\tau] + 2(\sin\tau - \tau)\right](\beta^* + \mathrm{i}\alpha)\Big] \\ &\quad \left. + \mathcal{O}\left(\frac{1}{r^3}\right)\right\}. \end{aligned} \tag{32}$$

Here, too, in the limit of high-field frequencies leading to equation (31), which also implies $r \gg 1$ and amounts to retaining only the first term in equation (32), the exponential term approaches unity, and so does the probability

$$P(\tau) = |\langle\beta(\tau)|\psi(\tau)\rangle|^2. \tag{33}$$

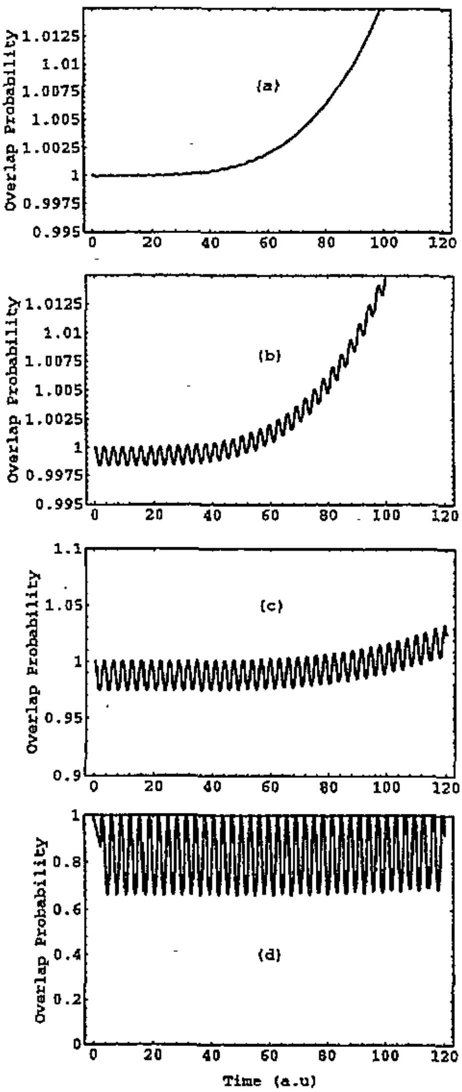

Figure 2. The overlap probability $|\langle\beta(t)|\psi(t)\rangle|^2$ shown here over approximately 19 field cycles and for peak field strengths, field frequency and oscillator natural frequency similar to figure 1.

In the rest of this paper, $P(\tau)$ will be referred to as the *overlap probability*. Acceptability of the approximate solution will depend on how close to unity this probability is. We show $P(t)$ over about 19 field cycles in figure 2.

As has been remarked by Frasca [2], the technique employed in this work is not limited to strong perturbations. It is also potentially applicable to the treatment of model and real atoms and ions interacting with fields having any strength. On the other hand, the series (2) may turn out to be neither analytical nor convergent in the field strength. The choice of a large development parameter ϵ has been made in order to ensure convergence of the series giving the time-evolved wavefunction. In the analysis leading to equation (24), however, ϵ has been set equal to one, just like in non-relativistic, non-degenerate, time-independent perturbation theory [8]. Thus, convergence of the series in the field strength may no longer be guaranteed. Nevertheless, the main results, especially equation (24), seem to suggest that $\psi(x, t)$ does converge for values of the parameter $r > 1$, or equivalently, for field frequencies in excess of $\omega_0/2$, where ω_0 is the oscillator's natural frequency. It must be maintained, of course, that no clear decision on convergence of the series, or lack of it, may be based on just three terms.

For high field intensities [11], the electron of a real atom leaves the interaction region, a spot of the order of a few micrometres, in typically 30 ps following ionization. With the effect of the binding potential diminishing, away from the centre of the interaction region, the electron is most suitably described by a Volkov state [12, 13]. Outside the interaction region, however, the electron travels towards the detector as a free particle. The evolution from the ground state to the Volkov state and finally to the free-particle state necessitates a clear specification of the time region over which a particular solution is claimed to hold. Many authors [14] have been reporting results calculated numerically from the evolution of the states of real and model atoms over typically ten to a few hundred field cycles. In these publications, the time region over which the calculation is made is important especially when pulsed laser fields are used. Field turn-on and turn-off effects have direct impact on the ensuing dynamics of the ejected electron [14]. In the case of a harmonic oscillator, an infinite number of evenly spaced bound states exists and no continuum is involved. Continued irradiation by laser light results in a continued climbing of the energy level ladder, as long as the oscillator is within the interaction region. Thus, the solution presented in this article holds from the instant the field is turned on at $t = 0$ until the oscillator has left the interaction region. If, however, the oscillator is somehow confined to the interaction region, then our solution should be taken to hold for time regions dictated by plots similar to the ones exhibited in figure 2.

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